Examples 4.2 – Horizontal and Vertical Asymptotes

1. Find any horizontal asymptotes of the following functions.

   (a) $f(x)=\dfrac{4x^2-2x+7}{3x^2+9x-1}$ (b) $h(t)=\dfrac{t^4+t^3+5t^2+5t}{2t^3-3t+4}$ (c) $g(x)=\dfrac{3x-7}{\sqrt{2x^2+4x}}$

   Solution: Horizontal asymptotes of a function can be found by evaluating limits at infinity and negative infinity. For a rational function, the terms with the highest powers in the numerator a denominator dominate the behavior, so all we need to consider are those terms.

   (a) Since $\lim\limits_{x\to+\infty} f(x)=\lim\limits_{x\to+\infty}\dfrac{4x^2}{3x^2}=\lim\limits_{x\to+\infty}\dfrac{4}{3}=\dfrac{4}{3}$, and $\lim\limits_{x\to-\infty} f(x)=\lim\limits_{x\to-\infty}\dfrac{4x^2}{3x^2}=\lim\limits_{x\to-\infty}\dfrac{4}{3}=\dfrac{4}{3}$,
   f has one horizontal asymptote $y = 4/3$ (in both directions).

   (b) Since $\lim\limits_{t\to+\infty} h(t)=\lim\limits_{t\to+\infty}\dfrac{t^4}{2t^3}=\lim\limits_{t\to+\infty}\dfrac{t}{2}=+\infty$, and $\lim\limits_{t\to-\infty} h(t)=\lim\limits_{t\to-\infty}\dfrac{t^4}{2t^3}=\lim\limits_{t\to-\infty}\dfrac{t}{2}=-\infty$,
   h has no horizontal asymptotes.

   (c) Although g is not rational, we can still analyze the terms with the highest powers, but we must be sure to keep the radical. In this example, we can use the fact that $\sqrt{x^2}=\pm x$, depending on whether x is positive or negative, and we find two horizontal asymptotes:

   $$\lim_{x\to+\infty} g(x)=\lim_{x\to+\infty}\frac{3x}{\sqrt{2x^2}}=\lim_{x\to+\infty}\frac{3x}{\sqrt{2}\sqrt{x^2}}=\lim_{x\to+\infty}\frac{3x}{\sqrt{2}(x)}=\lim_{x\to+\infty}\frac{3}{\sqrt{2}}=\frac{3}{\sqrt{2}}$$
   $$\lim_{x\to-\infty} g(x)=\lim_{x\to-\infty}\frac{3x}{\sqrt{2x^2}}=\lim_{x\to-\infty}\frac{3x}{\sqrt{2}\sqrt{x^2}}=\lim_{x\to-\infty}\frac{3x}{\sqrt{2}(-x)}=\lim_{x\to-\infty}\frac{3}{-\sqrt{2}}=-\frac{3}{\sqrt{2}}$$

2. Analyze the behavior on either side of the vertical asymptote $x = 1$ of the function from Example 4.1.1, $f(x)=\frac{x^2-x-2}{3x^2-9x+6}=\frac{(x+1)(x-2)}{3(x-1)(x-2)}$. Then view the graph of f near $x = 1$.

   Solution: Since f grows without bound ($+\infty$ or $-\infty$) as x approaches the asymptote, all we need to do is analyze the sign of f on either side of and close to $x = 1$. Because we are away from the hole at $x = 2$, we could cancel the common factors in the limits. Now, as $x\to1^-$, it follows that $x < 1$, hence $x - 1 < 0$. The opposite is true as $x\to1^+$. Therefore,

   $$\lim_{x\to1^-} f(x)=\lim_{x\to1^-}\frac{(x+1)}{3(x-1)}\left(\frac{(+)}{(-)}\right)=-\infty \quad\text{and}\quad \lim_{x\to1^+} f(x)=\lim_{x\to1^+}\frac{(x+1)}{3(x-1)}\left(\frac{(+)}{(+)}\right)=+\infty.$$