## Activity 4.2†‡ – Horizontal and Vertical Asymptotes

**FOR DISCUSSION**: *What is the purpose for analyzing one-sided limits?*
*Under what condition does a two-sided limit exist?*
*Under what conditions does a two-limit not exist?*
*How can we use limits to detect horizontal and vertical asymptotes?*

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1. When finding horizontal asymptotes of a function, we must set up and evaluate limits at infinity and negative infinity. Find the horizontal asymptotes, if any, of the given function. (**HINT**: You need only consider the terms with the highest powers in the numerator and denominator.)

$$f(x) = \frac{9x^4 - 6x^3 + 2x}{2x^4 + 7x - 1}$$

2. Compute the following limits at infinity. Be sure to write "lim" in each step until you actually compute the limit.

(a) $\lim\limits_{x \to +\infty} \dfrac{5x + x^2}{6 - 4x^2} =$

(b) $\lim\limits_{x \to -\infty} \dfrac{x^5 - 10x^2}{3x^6 + x^4 + 1} =$

(c) $\lim\limits_{x \to +\infty} \dfrac{2x + x^2}{81 - x} =$

(d) $\lim\limits_{x \to -\infty} \dfrac{\sqrt{x^2 + 1}}{3x - 4} =$

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† This activity is referenced in Lesson 4.3.
‡ This activity has supplemental exercises.

3. The function $f(x) = \dfrac{x^3 + 4x^2 + 5x + 2}{x^3 + 4x^2 - 3x - 18} = \dfrac{(x+1)^2(x+2)}{(x-2)(x+3)^2}$ has two vertical asymptotes, one at $x = 2$, and one at $x = -3$. In order to analyze the behavior near vertical asymptotes, we must set up and evaluate left- and right-hand limits.

(a) Analyze the behavior of $f$ near the vertical asymptote $x = 2$ by computing the following limits. Write DNE for a limit that does not exist.

$$\lim_{x \to 2^-} \frac{(x+1)^2(x+2)}{(x-2)(x+3)^2} =$$

$$\lim_{x \to 2^+} \frac{(x+1)^2(x+2)}{(x-2)(x+3)^2} =$$

$$\lim_{x \to 2} \frac{(x+1)^2(x+2)}{(x-2)(x+3)^2} =$$

(b) Analyze the behavior of $f$ near the vertical asymptote $x = -3$ by setting up and evaluating the appropriate limits (similar to Part (a)).

4. (a) Let $f(x) = \frac{15x^2}{x^2 - 100}$ . Compute $f'$ and list all points at which $f'$ is either zero or undefined. (Hint: Find the zeros of the numerator and the zeros of the denominator of $f'$. The points in this list that are also in the domain of $f$ are called "critical points.")

(b) Use the points from Part (a) and sign tests to find the intervals on which $f$ is increasing and the intervals on which $f$ is decreasing. Your answers must not include points where $f$ is undefined.

(c) Determine the behavior of $f$ on either side of the vertical asymptotes $x = 10$ and $x = -10$, but try do so by analyzing your sign tests in Part (b).

$\lim_{x \to -10^-} \frac{15x^2}{x^2 - 100} =$ $\qquad$ $\lim_{x \to -10^+} \frac{15x^2}{x^2 - 100} =$

$\lim_{x \to 10^-} \frac{15x^2}{x^2 - 100} =$ $\qquad$ $\lim_{x \to 10^+} \frac{15x^2}{x^2 - 100} =$

5. Use the graph of $f$ shown below to find the given limits. Write DNE if a limit does not exist.

(a) $\lim_{x \to -5^-} f(x) =$ ______

(b) $\lim_{x \to -5^+} f(x) =$ ______

(c) $\lim_{x \to -5} f(x) =$ ______

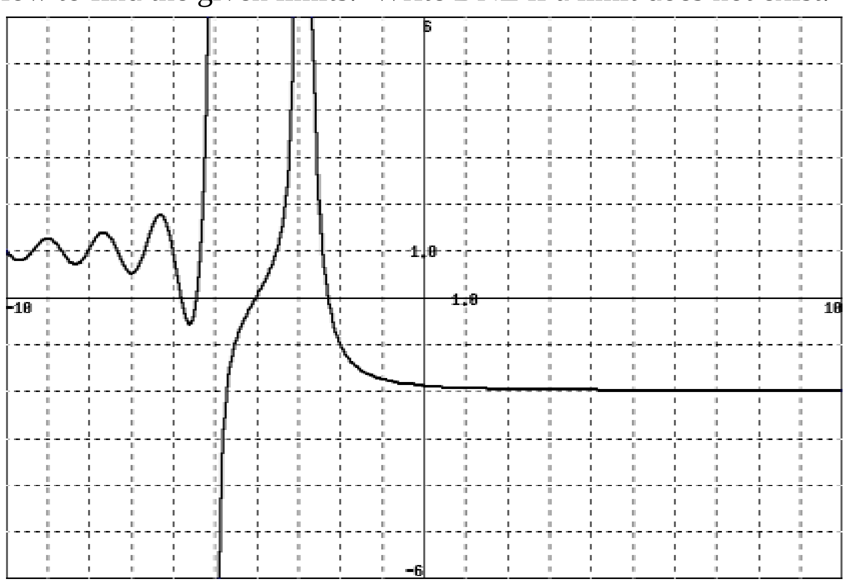
(d) $\lim_{x \to -3^-} f(x) =$ ______

(e) $\lim_{x \to -3^+} f(x) =$ ______

(f) $\lim_{x \to -3} f(x) =$ ______

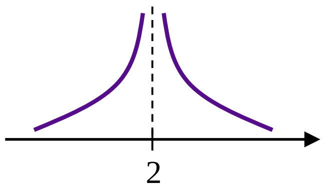
(g) $\lim_{x \to -\infty} f(x) =$ ______

(h) $\lim_{x \to +\infty} f(x) =$ ______

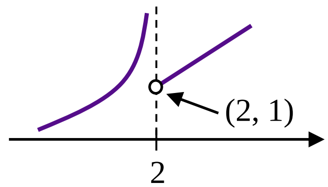


6. Intuitively, a function is **continuous** if its graph has no breaks, jumps, or holes. In this part, we will prepare ourselves for the mathematical definition of **continuity at a point**.

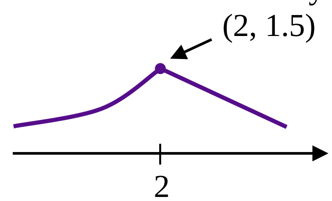
(a) For each of the following functions, evaluate $y(2)$, if possible. Write DNE if necessary.




$y(2) =$ ____________



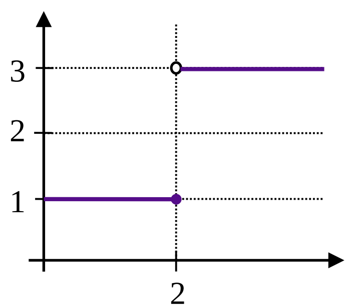

$y(2) =$ ____________




$y(2) =$ ____________

(b) Based on your observations from Part (a), what do you think must be true about $x = 2$ so that a function is continuous there? In other words, what must be true about $y(2)$?

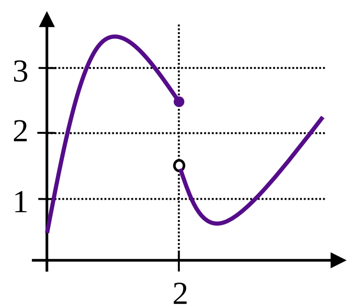
(c) For each of the following functions, graphically evaluate the limits at $x = 2$.




$\lim_{x \to 2^-} y(x) =$ ____________

$\lim_{x \to 2^+} y(x) =$ ____________

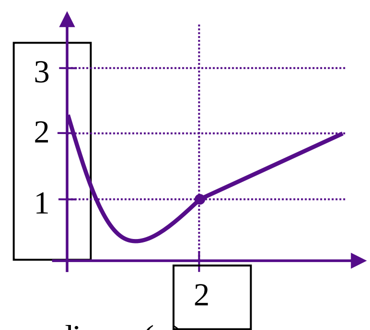
$\lim_{x \to 2} y(x) =$ ____________




$\lim_{x \to 2^-} y(x) =$ ____________

$\lim_{x \to 2^+} y(x) =$ ____________

$\lim_{x \to 2} y(x) =$ ____________



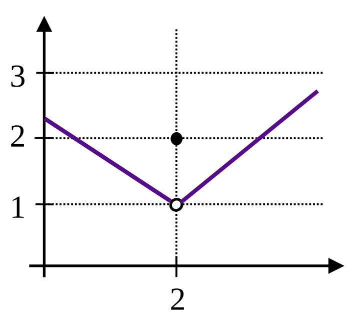

$\lim_{x \to 2^-} y(x) =$ ____________

$\lim_{x \to 2^+} y(x) =$ ____________

$\lim_{x \to 2} y(x) =$ ____________

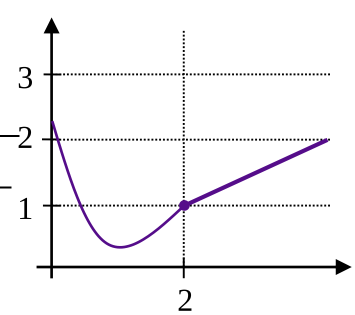
(d) Based on your answers to Part (c), what do you think must be true about the two-sided limit at $x = 2$ so that a function is continuous there?

(e) For the following functions, graphically evaluate $y(2)$ and the two-sided limit at $x = 2$.




$y(2) =$ ______

$\lim_{x \to 2} y(x) =$ ______




$y(2) =$ ______

$\lim_{x \to 2} y(x) =$ ______

(b) Based on your answers to Part (e), what do you think must be true about the function value and the limit at $x = 2$ so that a function is continuous there?