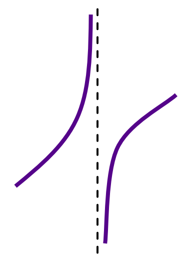
Lesson 4.1 – Analyzing Rational Functions

A **rational function** is a quotient function of the form $y = f(x) = \frac{N(x)}{D(x)}$ in which the numerator N and the denominator D are polynomials. A rational function is **proper** if the degree of N is smaller than the degree of D; otherwise it is **improper**.

Rewriting an improper rational function: Using long division, an improper rational function can be rewritten in the form $\frac{N(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$, where Q is the quotient, R is the remainder, and the rational function $\frac{R(x)}{D(x)}$ is proper.

Domain: The set of all real numbers except those that make $D(x) = 0$.

Graph: May have holes, **vertical asymptotes**, or both. In other words, a rational function is continuous on its domain and discontinuous elsewhere. Intuitively, a vertical asymptote can be thought of as a vertical line that the graph neither touches nor crosses. Moreover, the graph increases or decreases without bound as the graph approaches the asymptote from either side. We will discuss asymptotes in depth in the next lesson.



***y*-intercept**: Plug in $x = 0$, if possible, and solve for $y = f(0)$.

***x*-intercepts, holes, vertical asymptotes**: Factor N and D, but do not cancel common factors! Then use the zeros of N and D to detect x-intercepts, holes, and vertical asymptotes, if any. Some justification for these facts will come later in this chapter.

1. $N(a) = 0$ and $D(a) \neq 0 \;\rightarrow\; \frac{N(x)}{D(x)}$ has an ***x*-intercept** at $x = a$
2. $N(a) \neq 0$ and $D(a) = 0 \;\rightarrow\; \frac{N(x)}{D(x)}$ has a **vertical asymptote** at $x = a$
3. $N(a) = 0$ and $D(a) = 0 \;\rightarrow\; \frac{N(x)}{D(x)}$ has a **hole** at $x = a$ if the factor $(x - a)$ appears at least as many times in N as in D,

   or $\frac{N(x)}{D(x)}$ has a **vertical asymptote** at $x = a$ if the factor $(x - a)$ appears more times in D than in N.

Derivative: One option is to rewrite the quotient $\frac{N(x)}{D(x)}$ as a product $N(x) \cdot D(x)^{-1}$, and then use the product and chain rules. Another option is to use the so-called **quotient rule**, which you will derive in Activity 4.1.