Examples 4.1 – Analyzing Rational Functions

1. Consider the rational function $f(x) = \dfrac{x^2 - x - 2}{3x^2 - 9x + 6}$, and note that f is improper. Find each of the following:

   (i) Domain
   (ii) y-intercept
   (iii) x-intercepts, holes, vertical asymptotes
   (iv) A proper form using long division.

   Solution: Factor the numerator and denominator to get $\frac{x^2-x-2}{3x^2-9x+6} = \frac{(x+1)(x-2)}{3(x-1)(x-2)}$.

   (i) Since the denominator is zero at $x = 1$ and $x = 2$, we must exclude these numbers from the domain. We can express the domain in **set notation** as $\{x \mid x \neq 1, 2\}$, or in **interval notation** as $(-\infty, 1) \cup (1, 2) \cup (2, +\infty)$.

   (ii) The y-intercept is $f(0) = -2/6 = -1/3$.

   (iii) According to the factorization, f has an x-intercept at $x = -1$, a vertical asymptote at $x = 1$, and a hole at $x = 2$. To find the y-coordinate of the hole, cancel the common $(x - 2)$ and plug in $x = 2$ to what remains. In this case, the hole is at the point (2, 1).

   (iv) We can use long division to rewrite f as $f(x) = \frac{x^2-x-2}{3x^2-9x+6} = \frac{1}{3} + \frac{2x-4}{3x^2-9x+6}$.

2. Repeat Parts (i), (ii), and (iii) from Part 1 for the function $g(x) = \dfrac{\sqrt{x-2}}{x^2 - 9}$. (Even though g is not a rational function, a similar analysis can be done.)

   Solution: Factor the denominator to get $\frac{\sqrt{x-2}}{x^2-9} = \frac{\sqrt{x-2}}{(x+3)(x-3)}$.

   (i) Since the denominator is zero at $x = 3$ and $x = -3$, these numbers are not in the domain. Furthermore, the numerator requires that $x - 2 \geq 0$, hence $x \geq 2$. After taking these restrictions into account, we can express the domain in **set notation** as $\{x \mid x \geq 2$ and $x \neq 3\}$ or in **interval notation** as $[2, 3) \cup (3, +\infty)$.

   (ii) There is no y-intercept since we cannot plug in $x = 0$.

   (iii) The only zero of the numerator is $x = 2$, and the zeros of the denominator are $x = 3$ and $x = -3$. It follows that g has an x-intercept at $x = 2$ and a vertical asymptote at $x = 3$. (Since $x = -3$ is well outside the domain of g, the graph cannot have an asymptote there.)