Activity 4.1[1,†,‡] – Analyzing Rational Functions

FOR DISCUSSION: *What is a rational function? How can you tell if it is proper or improper? Briefly describe how to find the following for a rational function: Domain, vertical asymptotes, x-intercepts, holes, y-intercept.*

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1. Let $f(x) = \dfrac{x^3 - 5x^2 + 7x - 3}{x^2 - 1}$, and let $N(x) = x^3 - 5x^2 + 7x - 3$ be the numerator of f.

   (a) Verify that the numerator of f is zero at $x = 1$. That, verify that $N(1) = 0$.

   (b) Since $x = 1$ is a zero of $N(x)$, it follows that $(x - 1)$ is a factor of $N(x)$. In particular, $N(x) = (x - 1)g(x)$. Using long division, divide $N(x)$ by $(x - 1)$ to find the quadratic $g(x)$.

   (c) Now factor the quadratic $g(x)$ to get a complete factorization of the numerator, $N(x)$.

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[1] This activity contains new content.

† This activity is referenced in Lesson 4.1.

‡ This activity has supplemental exercises.

(d) Rewrite $f(x)=\dfrac{x^3-5x^2+7x-3}{x^2-1}$ so that the numerator and denominator are completely factored. DO NOT cancel common factors!

$$f(x)=\frac{x^3-5x^2+7x-3}{x^2-1}=$$

(e) The domain of f is the set of all real numbers except x = __________ .

(f) The x-intercept(s) of f is/are at x = __________ .

(g) The y-intercept of f is at y = __________ .

(h) The vertical asymptote(s) of f is/are at x = __________ .

(i) The hole(s) in the graph of f is/are at x = __________ .

(j) Since the degree of the numerator of f is greater than or equal to the degree of the denominator, f is an improper rational function. Use long division to write $f(x)$ in proper form $f(x)=Q(x)+\frac{R(x)}{D(x)}$.

2. We want a rule for computing the derivative of a quotient function $y = \frac{f}{g}$. To find one, we rewrite the quotient as a product $\frac{f}{g} = f \cdot \frac{1}{g}$, and then use the product and reciprocal rules:

$$\left(\frac{f}{g}\right)' = \left(f \cdot \frac{1}{g}\right)' = f' \cdot \frac{1}{g} + f \cdot \left(\frac{1}{g}\right)'$$

Since the derivative of 1/g requires the chain rule, we have

$$\left(\frac{f}{g}\right)' = f' \cdot \frac{1}{g} + f \cdot \left(\frac{-1}{g^2} \cdot g'\right) = \frac{f'}{g} - \frac{f \cdot g'}{g^2}$$

Now we combine the two terms on the right using the common denominator g^2:

$$\left(\frac{f}{g}\right)' = \frac{f'}{g} - \frac{f \cdot g'}{g^2} = \frac{f' \cdot g}{g^2} - \frac{f \cdot g'}{g^2} = \frac{f' \cdot g - f \cdot g'}{g^2}$$

We have just derived the **quotient rule**. You must memorize this rule!

Quotient rule: $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2}$

Note that the numerator is the same as the product rule except with a minus sign. Be careful not to mix up the terms in the numerator, otherwise the sign of the derivative will be incorrect.

Example: Compute the derivative of $f(x) = \frac{x^3 - 5x^2 + 7x - 3}{x^2 - 1}$ using the quotient rule.

Solution:

$$\begin{aligned} f'(x) &= \frac{(x^3 - 5x^2 + 7x - 3)'(x^2 - 1) - (x^3 - 5x^2 + 7x - 3)(x^2 - 1)'}{(x^2 - 1)^2} \\ &= \frac{(3x^2 - 10x + 7)(x^2 - 1) - (x^3 - 5x^2 + 7x - 3)(2x)}{(x^2 - 1)^2} \\ &= \frac{x^4 - 10x^2 + 16x - 7}{(x^2 - 1)^2} \end{aligned}$$

3. Use the quotient rule to find the derivative of each function.

(a) $y = \dfrac{x}{2x-3}$

(b) $y = \dfrac{x^2+3x}{x+4}$

(c) $y = \dfrac{x^2-2}{3x^2-9x+6}$

(d) $g(x) = \dfrac{\sqrt{x-2}}{x^2-9}$

4. Find the extrema, if any, of the function in 3(b).

5. A drug is injected into the bloodstream of a patient. The concentration (in mg/cm^3) of the drug in the bloodstream t hours after the injection is $C(t) = \dfrac{0.16t}{t^2 + 6}$.

(a) What is the concentration of the drug in the bloodstream after 3 hours?

(b) Use the quotient rule to determine the rate of change of the drug concentration after t hours. Simplify your answer.

(c) Determine how fast the concentration of the drug is changing after 3 hours.