Lesson 3.6 – Integrals of Polynomials

The power rule for differentiating x^n is a two step process:

(1) Multiply by the old power of x;
(2) Subtract 1 from the old power to get the new power of x.

$$\frac{d}{dx}\left(x^n\right) = n \cdot x^{n-1}$$

The power rule for integrating x^n is the two step process that undoes the steps for differentiation:

(1) Add 1 to the old power to get the new power of x;
(2) Divide by the new power of x.

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

Unfortunately, this process results in division by zero when applied to x^{-1}, so we will consider this very special case later.

Power rule (for integration): If $n \neq -1$, then $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C = \frac{1}{n+1} \cdot x^{n+1} + C$.

Before we can describe how to integrate a polynomial, we must state two important properties of the integral which can be derived from the corresponding differentiation properties.

General properties of the integral:

Constant Multiple Rule: If k is a constant, then $\int k \cdot f(x) dx = k \cdot \int f(x) dx$.

Sum/Difference Rule: $\int (f(x) \pm g(x)) \, dx = \int f(x) dx \pm \int g(x) dx$.

It follows that the indefinite integration of a polynomial (or any sum/difference of powers) consists of integrating term-by-term using the power rule and properties. Since the derivative of a non-constant nth-degree polynomial is a polynomial having degree $n - 1$, the integral of a nonzero nth-degree polynomial is a polynomial having degree $n + 1$.

For definite integrals, the area calculations in verifying the FTC for constant and linear functions were based on area formulas for a rectangle and a trapezoid, respectively. The problem with finding the area bounded by an arbitrary polynomial function is that no geometrical formula exists for finding such an area. In Activity 2.6, we introduced a technique for finding the exact area by taking better and better approximations using areas of rectangles. We will formalize this idea in Lesson 8.2, but for now we state the fundamental theorem for polynomials.

Fundamental Theorem of Calculus (for polynomial functions): If $f'(x)$ is an n^{th} degree polynomial, then the net (signed) area bounded by the graph of $f'(x)$ on the interval $[x_0, x_1]$ is equal to the net change in any $(n+1)^{\text{st}}$ degree antiderivative $f(x)$ on $[x_0, x_1]$. That is,

$$\int_{x_0}^{x_1} f'(x) dx = f(x)\Big|_{x_0}^{x_1} = f(x_1) - f(x_0)$$