Examples 3.5 – Piecewise Functions

1. Discuss the continuity and differentiability of the function $f(x) = \begin{cases} x^2 - 6x + 6, & \text{if } x \le 2 \\ x + 1, & \text{if } x > 2 \end{cases}$.

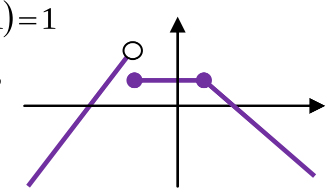
Solution: Note that the continuity and differentiability of *f* ultimately depends on what is happening at *x* = 2. For continuity, we need to check whether or not the function values are the same from either side of *x* = 2. Formally, we must check the **left-hand limit** (a "limit from the left") and the **right-hand limit** (a "limit from the right"):

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \left(x^2 - 6x + 6\right) = -2 \qquad \text{and} \qquad \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x + 1) = 3.$$

Since these values do not agree, *f* is not continuous at *x* = 2. This fact implies that the function is not differentiable at *x* = 2.

2. Discuss the continuity and differentiability of the function $g(x) = \begin{cases} 2x + 6, & \text{if } x < -2 \\ 1, & \text{if } -2 \le x < 1 \\ 2 - x, & \text{if } x \ge 1 \end{cases}$.

Solution: The domain of *g* has break points at *x* = –2 and at *x* = 1. The **one-sided limits** at *x* = –2 are $\lim_{x \to -2^-} (2x + 6) = 2$ and $\lim_{x \to -2^+} (1) = 1$, so *g* is neither continuous nor differentiable at *x* = –2. The one-sided limits at *x* = 1 are $\lim_{x \to 1^-} (1) = 1$ and $\lim_{x \to 1^+} (2 - x) = 1$. Since *g* is defined at *x* = 1, it is continuous there, but this does not mean it is differentiable. We must check derivatives on either side of *x* = 1:



$$\lim_{x \to 1^-} g'(x) = \lim_{x \to 1^-} (0) = 0 \qquad \text{and} \qquad \lim_{x \to 1^+} g'(x) = \lim_{x \to 1^+} (-1) = -1.$$

Since the slopes on either side of *x* = 1 are different, *g* has a sharp corner at *x* = 1, thus it is not differentiable there.

3. Find numbers *a* and *b* so that $f(x) = \begin{cases} x^2 - 6x + 6, & \text{if } x \le 2 \\ ax + b, & \text{if } x > 2 \end{cases}$ is differentiable everywhere.

Solution: We must have $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$ and $\lim_{x \to 2^-} f'(x) = \lim_{x \to 2^+} f'(x)$. For the first condition, we have $\lim_{x \to 2^-} \left(x^2 - 6x + 6\right) = -2$ and $\lim_{x \to 2^+} (ax + b) = 2a + b$, and for the second, $\lim_{x \to 2^-} (2x - 6) = -2$ and $\lim_{x \to 2^+} (a) = a$. Therefore, *a* = –2 and *b* = 2, and the function $f(x) = \begin{cases} x^2 - 6x + 6, & \text{if } x \le 2 \\ -2x + 2, & \text{if } x > 2 \end{cases}$ is differentiable (and continuous) everywhere.