Lesson 3.4 – Products of Functions

Recall, the derivative of a sum or difference of two functions is the sum or difference of the derivatives. This property does not hold for products, as we demonstrate with a simple example. Let $f(x) = x$ and $g(x) = x^2$, so that $f(x) \cdot g(x) = x \cdot x^2 = x^3$. On one hand, the derivative of the product is $(f(x) \cdot g(x))' = 3x^2$, and on the other hand, the product of the derivatives is $f'(x) \cdot g'(x) = 2x$. Thus, the derivative of a product is *not* the product of the derivatives. If an example is not convincing enough, then notice that the units will never match:

$$\text{Units for } (f(x) \cdot g(x))' = \frac{(\text{units for } f) \cdot (\text{units for } g)}{(\text{units for } x)}$$

$$\text{Units for } f'(x) \cdot g'(x) = \frac{(\text{units for } f)}{(\text{units for } x)} \cdot \frac{(\text{units for } g)}{(\text{units for } x)} = \frac{(\text{units for } f) \cdot (\text{units for } g)}{(\text{units for } x)^2}$$

Let f and g be functions defined at x such that $f'(x)$ and $g'(x)$ exist (i.e., the tangent lines to f and g exist at x), and let $\Delta x \neq 0$. Then the average rate of change of $f \cdot g$ is

$$\begin{aligned}
\frac{\Delta(f \cdot g)}{\Delta x} &= \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x} \\
&= \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x) + f(x)g(x+\Delta x) - f(x)g(x+\Delta x)}{\Delta x} \\
&= \frac{(f(x+\Delta x)g(x+\Delta x) - f(x)g(x+\Delta x)) + (f(x)g(x+\Delta x) - f(x)g(x))}{\Delta x} \\
&= \frac{(f(x+\Delta x) - f(x))g(x+\Delta x)}{\Delta x} + \frac{f(x)(g(x+\Delta x) - g(x))}{\Delta x} \\
&= \frac{f(x+\Delta x) - f(x)}{\Delta x} \cdot g(x+\Delta x) + f(x) \cdot \frac{g(x+\Delta x) - g(x)}{\Delta x} \\
&= \frac{\Delta f}{\Delta x} \cdot g(x+\Delta x) + f(x) \cdot \frac{\Delta g}{\Delta x}
\end{aligned}$$

As $\Delta x \to 0$, the left-hand side approaches $(f \cdot g)'(x)$, and the first and second terms on the right-hand side approach $f'(x) \cdot g(x)$ and $f(x) \cdot g'(x)$, respectively. Therefore, we have derived a rule for the rate of change of the product $f \cdot g$ for every x at which f and g have derivatives:

Product rule:

Leibniz notation: $\frac{d}{dx}(f \cdot g) = \frac{df}{dx} \cdot g(x) + f(x) \cdot \frac{dg}{dx}$

Prime notation: $(f \cdot g)(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$