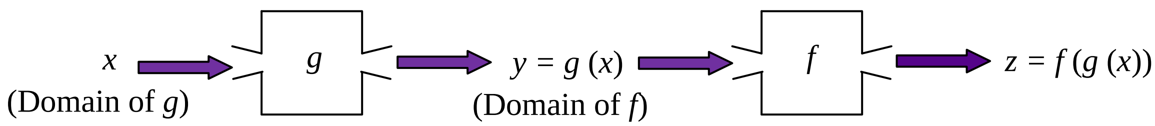
Lesson 3.3 – Composite Functions

Using basic rules for differentiation, we can find the rates of change of functions like $z(y)=\sqrt{y}$ and $y(x)=400x^2+2500$. Sometimes, however, we must compute the rate of change of more complicated functions such as $z(x)=\sqrt{400x^2+2500}$. Since this function is "composed" of a radical on the "outside" and a polynomial on the "inside," we are not able to apply the basic rules. We need a new rule for finding rates of change of these so-called **composite functions**.

If the outputs of a function $y=g(x)$ are in the domain of a function $z=f(y)$, then we can form the **composite function** $z=f(g(x))=(f\circ g)(x)$. Think of a composite as a *chain* of functions:



Sometimes it also helps to think of *g* as the **inside** function and *f* as the **outside** function so that $f(g(x))=OUTSIDE(INSIDE(x))$. In most cases, the "inside" function will appear inside parentheses, a radical, a denominator, or an exponent. Intuitively, a change in *x* produces a change in *y*, which in turn produces a change in *z*. The combined effect is that a change in *x* produces a change in *z*, but how, exactly? Let Δx be a nonzero change in *x*, let Δy be a nonzero change in *y* produced by Δx, and let Δz be a nonzero change in *z* produced by Δy. Then the average rate of change in the composite $z=(f\circ g)(x)$ with respect to *x* is

$$\frac{\Delta z}{\Delta x}=\frac{\Delta z}{\Delta y}\cdot\frac{\Delta y}{\Delta x}$$

As $\Delta x\to 0$, the left-hand side approaches $\frac{dz}{dx}$, and it seems reasonable to suspect that the right-hand side approaches the product $\frac{dz}{dy}\cdot\frac{dy}{dx}$.

Chain rule (for differentiating composite functions): Let $y=g(x)$ and suppose z is a composite function $=f(y)=f\big(g(x)\big)=OUTSIDE(INSIDE(x))$. The rate of change of z is as follows:

Leibniz notation: $$\frac{dz}{dx}=\frac{dz}{dy}\cdot\frac{dy}{dx}$$

Prime notation: $$(f\circ g)'(x)=f'(g(x))\cdot g(x)$$

Intuitively: $$\frac{d\,OUTSIDE}{dx}=\frac{d\,OUTSIDE}{d\,INSIDE}\cdot\frac{d\,INSIDE}{dx}$$