Examples 3.3 – Composite Functions

1. Suppose that the population of a town is $g(x) = 400x^2 + 2500$ people, where x is years after 2000, and that $f(g) = \sqrt{g}$ is the level of CO_2 pollution in the air in parts per million, where g is the population.

   (a) Complete the table of data.

   Solution:

x	$g(x) = 400x^2 + 2500$	$f(g(x)) = \sqrt{g(x)}$
2	4100	64
5	12500	112

   (b) Write down a model (with units) that represents the CO_2 level as a function of time. Find the rate of change model (with units), and then find how quickly the CO_2 level was changing in 2007.

   Solution: The function $(f \circ g)(x) = \sqrt{400x^2 + 2500}$ is the CO_2 level in parts per million, where x is years after 2000. The derivative of $f(g) = \sqrt{g}$ (the outside) is $f'(g) = \frac{1}{2\sqrt{g}}$, and the derivative of $g(x) = 400x^2 + 2500$ (the inside) is $g'(x) = 800x$. By the chain rule,

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x) = \frac{1}{2\sqrt{g(x)}} \cdot 800x = \frac{400x}{\sqrt{400x^2 + 2500}} \text{ ppm per year,}$$

   where x is years after 2000. In 2007, the level was increasing by $(f \circ g)'(7) \approx 19$ ppm/yr.

2. Find the derivative of each of the following:

   (a) $y = (x^2 - 3x)^5$ (b) $y = (f(x))^5$ (c) $y = \sqrt[3]{1 - 20x + 100x^4}$ (d) $y = \dfrac{7}{6\sqrt[5]{x^3 - 2x}}$

   Solution: (a) $y = (x^2 - 3x)^5 \quad \rightarrow \quad y' = 5(x^2 - 3x)^4 \cdot (2x - 3)$

   (b) $y = (f(x))^5 \quad \rightarrow \quad y' = 5(f(x))^4 \cdot f'(x)$

   (c) $y = (1 - 20x + 100x^4)^{1/3} \quad \rightarrow \quad y' = \frac{1}{3}(1 - 20x + 100x^4)^{-2/3} \cdot (-20 + 400x^3)$

   (d) $y = \frac{7}{6}(x^3 - 2x)^{-1/5} \quad \rightarrow \quad y' = -\frac{7}{30}(x^3 - 2x)^{-6/5} \cdot (3x^2 - 2)$