Examples 3.2 – Polynomial Functions

1. Find $f'(x)$ if $f(x) = -3x^7 + 4x^3 - 10x$.

   Solution: $f'(x) = \left(-3x^7 + 4x^3 - 10x\right)' = \left(-3x^7\right)' + \left(4x^3\right)' - \left(10x\right)' = -21x^6 + 12x^2 - 10$

2. Suppose the total amount of outstanding mortgage debt in the U.S. for years between 1980 and 2000 can be modeled by $A(t) = 0.173t^4 - 6.24t^3 + 71.06t^2$ billion dollars t years after 1980. Find $A(16)$ and $A'(16)$, and interpret the answers.

   Solution: First of all, $A(16)$ =3970.048, which means that the total amount of outstanding mortgage debt in 1996 was about 3970.048 billion dollars. Second, the rate of change of mortgage debt between 1980 and 2000 was $A'(t) = 0.692t^3 - 18.72t^2 + 142.12t$ billion dollars per year, where t is years after 1980. Therefore, $A'(16)$ = 316.032, which means that the total amount of outstanding mortgage debt in 1996 was increasing by about 316.032 billion dollars per year.

3. Investigate the end behavior of the functions from Examples 1 and 2.

   Solution: The leading term of a polynomial dictates end behavior. Therefore,

   $$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\left(-3x^7\right) = -\infty \qquad \text{and} \qquad \lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\left(-3x^7\right) = +\infty$$

   and

   $$\lim_{t\to+\infty} A(t) = \lim_{t\to+\infty}\left(0.173t^4\right) = +\infty \qquad \text{and} \qquad \lim_{t\to-\infty} A(t) = \lim_{t\to-\infty}\left(0.173t^4\right) = +\infty$$

4. Find the x-intercepts, critical points, and end behavior of $f(x) = x^5 - 4x^3 - 21x$.

   Solution: To find the x-intercepts, we set $f(x) = x^5 - 4x^3 - 21x = 0$. In this case,

   $$x^5 - 4x^3 - 21x = x(x^4 - 4x^2 - 21) = x(x^2 - 7)(x^2 + 3) = 0$$

   implies that $x = 0, \sqrt{7}$, and $-\sqrt{7}$. The derivative, $f'(x) = 5x^4 - 12x^2 - 21$, is a **quadratic form** such that $x^2 = \frac{12\pm\sqrt{564}}{10} = \frac{6\pm\sqrt{141}}{5}$, but $x^2 = \frac{6-\sqrt{141}}{5} \approx -1.17$ has no solution. Therefore, the critical points are $x = \pm\sqrt{\frac{6+\sqrt{141}}{5}} \approx \pm 1.891$. The end behavior is

   $$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\left(x^5\right) = +\infty \qquad \text{and} \qquad \lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\left(x^5\right) = -\infty$$