Examples 3.1 – Power Functions

1. Verify the power rule for $y = x^m$, where m is a positive integer.

Solution: Observe that

$$\begin{aligned}(x - x_0)\left(x^{m-1} + x_0 x^{m-2} + \cdots + x_0{}^{m-2} x + x_0{}^{m-1}\right) &= x^m + x_0 x^{m-1} + \cdots + x_0{}^{m-1} x \\ &\quad - x_0 x^{m-1} - \cdots - x_0{}^{m-1} x - x_0{}^m \\ &= x^m - x_0{}^m\end{aligned}$$

Therefore, $\dfrac{x^m - x_0{}^m}{x - x_0} = x^{m-1} + x_0 x^{m-2} + \cdots + x_0{}^{m-2} x + x_0{}^{m-1}$, and it follows that

$$\left.\frac{d}{dx}\left(x^m\right)\right|_{x=x_0} = \lim_{x \to x_0} \frac{x^m - x_0^m}{x - x_0} = \lim_{x \to x_0}\Bigg(\underbrace{x^{m-1} + x_0 x^{m-2} + \cdots + x_0{}^{m-2} x + x_0{}^{m-1}}_{(m \text{ terms})}\Bigg) = m x_0{}^{m-1}$$

2. Compute each of the following derivatives:

(a) $\dfrac{d}{dx}\left(x^5\right)$ (b) $\dfrac{d}{dx}\left(\dfrac{1}{x}\right)$ (c) $\dfrac{d}{dx}\left(\sqrt[3]{x^2}\right)$ (d) $\dfrac{d}{dx}\left(\dfrac{2}{x\sqrt{x}} - \dfrac{3}{\sqrt[3]{x}}\right)$

Solution: Note that converting radicals and denominators to exponents is an important step.

(a) $\dfrac{d}{dx}\left(x^5\right) = 5x^4$

(b) $\dfrac{d}{dx}\left(\dfrac{1}{x}\right) = \dfrac{d}{dx}\left(x^{-1}\right) = -x^{-2} = -\dfrac{1}{x^2}$

(c) $\dfrac{d}{dx}\left(\sqrt[3]{x^2}\right) = \dfrac{d}{dx}\left(x^{2/3}\right) = \dfrac{2}{3} x^{-1/3} = \dfrac{2}{3\sqrt[3]{x}}$

(d) $\dfrac{d}{dx}\left(\dfrac{2}{x\sqrt{x}} - \dfrac{3}{\sqrt[3]{x}}\right) = \dfrac{d}{dx}\left(2x^{-3/2} - 3x^{-1/3}\right) = -3x^{-5/2} + x^{-4/3} = -\dfrac{3}{x^2\sqrt{x}} + \dfrac{1}{x\sqrt[3]{x}}$

3. According to Newton's law of universal gravitation, two masses m_1 and m_2 (in kg) at a distance of r meters apart attract each other with a force of $F(r) = G\dfrac{m_1 m_2}{r^2}$ Newtons, where G is the gravitational constant. Find $F'(r)$.

Solution: If $F(r) = G\dfrac{m_1 m_2}{r^2} = G m_1 m_2 r^{-2}$ N, then $F'(r) = -2G m_1 m_2 r^{-3} = -2G\dfrac{m_1 m_2}{r^3}$ N/m.