Lesson 2.6 – Integrals of Linear and Quadratic Functions

In this chapter, we found that the derivative of a quadratic is linear, and the derivative of a cubic is quadratic. It follows that the antiderivatives of a linear function are quadratic and the antiderivatives of a quadratic function are cubic:

$$\text{If } \left(\tfrac{1}{2}ax^2 + bx + c\right)' = ax + b, \text{ then } \int (ax+b)\,dx = \tfrac{1}{2}ax^2 + bx + C.$$

$$\text{If } \left(\tfrac{1}{3}ax^3 + \tfrac{1}{2}bx^2 + cx + d\right)' = ax^2 + bx + c, \text{ then } \int (ax^2+bx+c)\,dx = \tfrac{1}{3}ax^3 + \tfrac{1}{2}bx^2 + cx + D.$$

We know from Lesson 1.4 that the net area bounded by a constant function on an interval is the same as the net change in any of its linear antiderivatives on that interval:

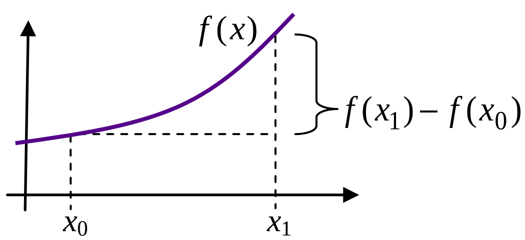
$$\int_{x_0}^{x_1} m\,dx = (mx)\Big|_{x_0}^{x_1} = mx_1 - mx_0$$

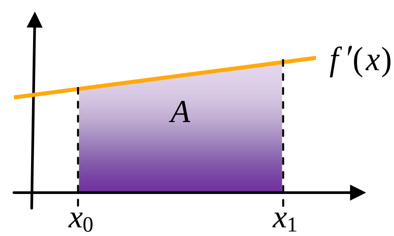
We stated this result as the Fundamental Theorem of Calculus for constant functions. Now we will show that the theorem is true for linear functions. Quadratics will need to wait, however.

Observation 1: If f has a linear rate of change $f'(x) = mx + b$ on $[x_0, x_1]$, then an antiderivative of f' is $f(x) = \tfrac{1}{2}mx^2 + bx + C$ for some constant C. The net change in f on $[x_0, x_1]$ is

$$f(x_1) - f(x_0) = \left(\tfrac{1}{2}mx_1^{\,2} + bx_1 + C\right) - \left(\tfrac{1}{2}mx_0^{\,2} + bx_0 + C\right) = \tfrac{1}{2}m(x_1 + x_0)\Delta x + b\Delta x.$$

Observation 2: Let A denote the net (signed) area bounded by f' on $[x_0, x_1]$. From geometry, the area of this trapezoid is $A = \tfrac{1}{2}\big((mx_1 + b) + (mx_0 + b)\big)\Delta x = \tfrac{1}{2}m(x_1 + x_0)\Delta x + b\Delta x$.





Therefore, the net area bounded by a linear function on an interval is the same as the net change in any of its quadratic antiderivatives on that interval.

Fundamental Theorem of Calculus (for linear functions): If $f'(x)$ is a linear function, then the net (signed) area bounded by the graph of $f'(x)$ on the interval $[x_0, x_1]$ is equal to the net change in any quadratic antiderivative $f(x)$ on $[x_0, x_1]$. That is,

$$\int_{x_0}^{x_1} f'(x)\,dx = f(x)\Big|_{x_0}^{x_1} = f(x_1) - f(x_0)$$