Activity 2.6[1†‡] – Integrals of Linear and Quadratic Functions

FOR DISCUSSION: *What is the indefinite integral of ax + b? What about* $ax^2 + bx + c$*?*
Explain the FTC (for linear functions) in your own words.

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1. Evaluate the indefinite integrals.

   (a) $\int 10x\,dx =$

   (b) $\int (3t+4)\,dt =$

   (c) $\int (1-u)\,du =$

2. Evaluate the definite integrals. (**HINT**: Use your answers from Problem 1.)

   (a) $\int_1^2 10x\,dx =$

   (b) $\int_0^1 (3t+4)\,dt =$

   (c) $\int_{-2}^3 (1-u)\,du =$

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[1] This activity contains new content.
[†] This activity is referenced in Lesson 3.6, Examples 3.6, and Lesson 5.6.
[‡] This activity has supplemental exercises.

3. Evaluate the indefinite integrals.

(a) $\int \left(2x^2 - x + 7\right)dx =$

(b) $\int \left(32 - 6w + w^2\right)dw =$

4. A population of cattle is increasing at a rate of $P'(t) = 10 + 8t$ cows per year, where t is measured in years. By how much does the population increase between the 5^{th} and 8^{th} years?

5. A stone is thrown straight up from the edge of a roof, 175 ft above the ground, at a speed of 16 ft/s. Recall, the acceleration due to gravity is –32 ft/s^2.

(a) Find the equation for the stone's velocity, *v*, using integration and the initial velocity.

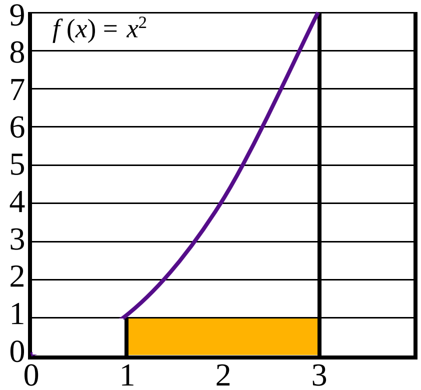
(b) Find the equation for the stone's position, *s*, above the ground using integration and the initial position.

(c) At what time does the stone hit the ground? (**HINT**: When was the position equal to 0?)

(d) What was the velocity of the stone at the moment it hit the ground?

The area calculations in verifying the FTC for constant and linear functions were based on area formulas for a rectangle and a trapezoid, respectively. The problem with finding the area bounded by a quadratic function, for example, is that no geometrical formula exists for finding such an area. For now, we can use areas of rectangles to get an approximation of the exact area.

Suppose we want to approximate $\int_1^3 x^2 dx$, the area beneath the graph of $f(x) = x^2$ on [1, 3].

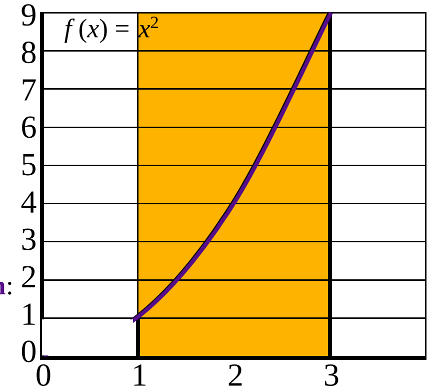


A rectangle of width $\Delta x = 2$ and height $f(1) = 1$ (at left endpoint) gives a **left-hand approximation**:

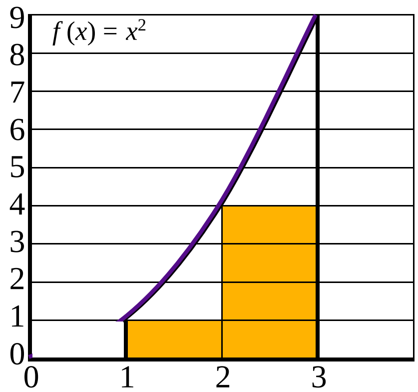
$$L_1 = f(1) \cdot \Delta x = 1 \cdot 2 = 2$$

A rectangle of width $\Delta x = 2$ and height $f(3) = 9$ (at right endpoint) gives a **right-hand approximation**:

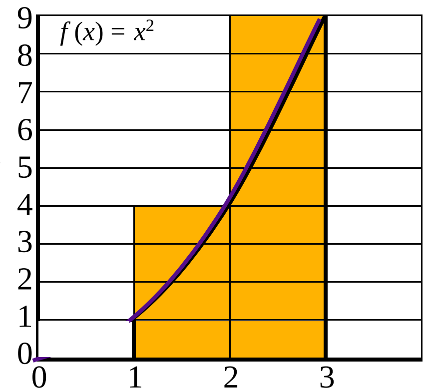
$$R_1 = f(3) \cdot \Delta x = 9 \cdot 2 = 18$$



We find that $2 \le \int_1^3 x^2 dx \le 18$, but this is a poor approximation.



Let us try to approximate the area using 2 rectangles of width $\Delta x = 1$. in this case, the left- and right-hand approximations are determined by the left- and right-hand endpoints of each subinterval.



For the left-hand approximation, the heights are $f(1) = 1$ and $f(2) = 4$. Therefore,

$$\begin{aligned} L_2 = f(1)\cdot\Delta x + f(2)\cdot\Delta x &= (f(1) + f(2))\cdot\Delta x \\ &= (1 + 4) \cdot 1 \\ &= 5 \end{aligned}$$

For the right-hand approximation, the heights are $f(2) = 4$ and $f(3) = 9$. Therefore,

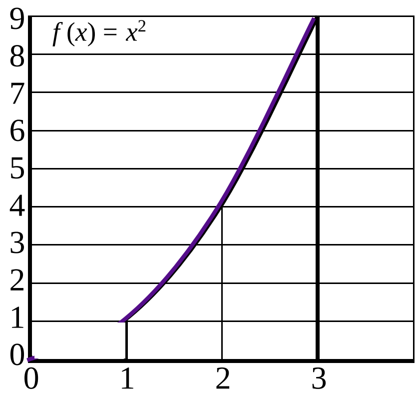
$$\begin{aligned} R_2 = f(2)\cdot\Delta x + f(3)\cdot\Delta x &= (f(2) + f(3))\cdot\Delta x \\ &= (4 + 9) \cdot 1 \\ &= 13 \end{aligned}$$

We find that $5 \le \int_1^3 x^2 dx \le 13$, and this is a better approximation.

It looks like more rectangles will give a better approximation. Let Δx denote the width of each rectangle, and suppose we use 4 rectangles on the interval [1, 3]. Then each would have width

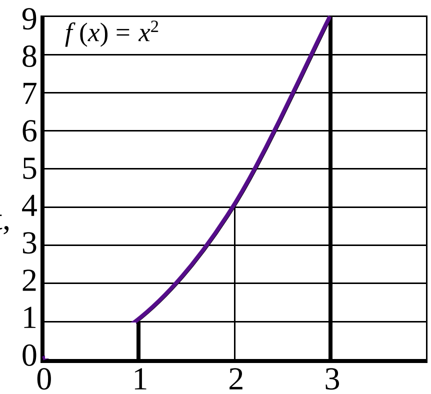
$$\Delta x = \frac{3-1}{4} = \frac{1}{2}$$

The height of each rectangle depends on whether we use left- or right-hand endpoints.



6. On the graph to the left, draw 4 rectangles of width $\Delta x = 1/2$ whose heights are at left-hand endpoints.

7. On the graph to the right, draw 4 rectangles of width $\Delta x = 1/2$ whose heights are at right-hand endpoints.



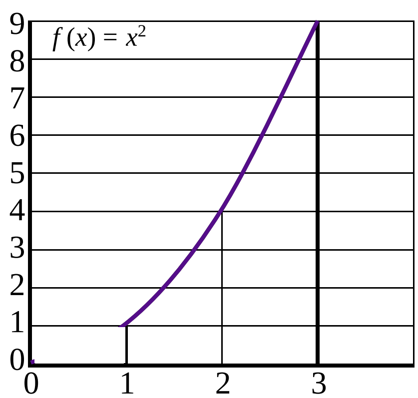
8. Now compute the left- and right-hand approximations for 4 rectangles. Since each area contains Δx, it can be factored out. Enter consecutive heights in the parentheses, with the common Δx multiplied at the end.

$$L_4 = (\_\_\_\_\_\_\_\_ + \_\_\_\_\_\_\_\_ + \_\_\_\_\_\_\_\_ + \_\_\_\_\_\_\_\_) \cdot \_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_$$

$$R_4 = (\_\_\_\_\_\_\_\_ + \_\_\_\_\_\_\_\_ + \_\_\_\_\_\_\_\_ + \_\_\_\_\_\_\_\_) \cdot \_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_$$

We find that ________ $\le \int_1^3 x^2 dx \le$ ________, and this is an even better approximation.

9. Repeat steps 1 and 2 on the graphs below for 8 rectangles. In this case, Δx = ________.

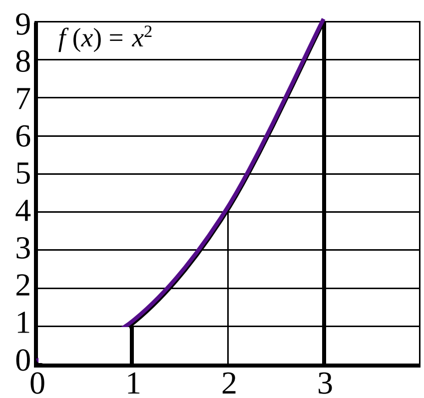


If you wish, verify that

$L_8 = 7.6875$

and

$R_8 = 9.6875$.



10. Notice that the more rectangles there are, the closer their tops are to the graph. Also observe how the left- and right-hand approximations are getting closer and closer to the exact area of 26/3 (≈ 8.667). We will be able to compute the exact area later.