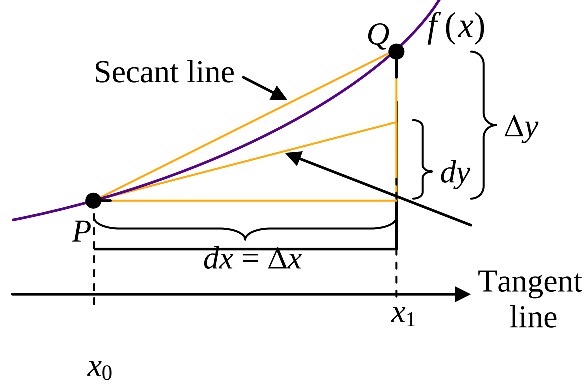
Lesson 2.5 – Linear Approximation

Suppose P and Q are distinct points on the graph of f. Then the slope of the secant line between P and Q is an approximation of the slope of the tangent line at P, if one exists. In symbols, $\frac{\Delta y}{\Delta x} \approx f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$. Since $f'(x)$ is a slope, it is convenient to express it in "rise-over-run" form without having to use limit notation. Because we already use Δx and Δy to denote the run and rise, respectively, of the secant line on the interval between P and Q, we use the **differentials** dx and dy to denote the run and rise in the tangent line on the interval. It follows that the slope of the tangent line is $\frac{dy}{dx}$, a rise over run! This notation is called **Leibniz notation**.



Leibniz notation for derivative functions:

$$\frac{dy}{dx} = f'(x), \quad \frac{d^2y}{dx^2} = f''(x), \quad \frac{d^3y}{dx^3} = f'''(x) \text{ , etc.}$$

Leibniz notation for derivative at a point:

$$\left.\frac{dy}{dx}\right|_{x=x_0} = f'(x_0)$$

Differential operator notation: $\frac{d}{dx}\big(f(x)\big) = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

In the same way that we used the slope of a secant line to approximate the slope of a tangent line, we can use a tangent line to approximate a nearby function value. This technique is called **linear approximation**, and it works because a well-behaved graph will resemble its tangent line if we zoom in close enough (Recall Activity 2.1). The tangent line at the point $P = (x_0, f(x_0))$ has slope $f'(x_0)$ and point-slope equation

$$y - f(x_0) = f'(x_0)(x - x_0) \qquad \text{or} \qquad y = f(x_0) + f'(x_0)(x - x_0)$$

By moving along this tangent line at P, we can approximate an unknown function value near P, say at $Q = (x_1, f(x_1))$, as

$$f(x_1) \approx f(x_0) + f'(x_0)(x_1 - x_0)$$

Linear approximation can be used to analyze error in physical measurements. Suppose the exact value of some quantity is x_0, but we measure it as x. That is, the error in measurement is Δx. If we use a function f to perform a calculation with the erroneous x, then the error in $f(x)$, which we are calling Δy, is propagated:

Propagated error: $\Delta y \approx dy = f'(x)dx = f'(x)\Delta x$

Relative (percentage) error: $\frac{\Delta y}{y} \approx \frac{dy}{y} = \frac{f'(x)dx}{f(x)}$