Examples 2.5 – Linear Approximation

1. Use Leibniz notation to find the first three derivatives of $y = 11x^2 - 4x + 7$.

   Solution: We have $\frac{dy}{dx} = 22x - 4$, $\frac{d^2y}{dx^2} = 22$, and $\frac{d^3y}{dx^3} = 0$.

2. Use differential operator notation to find the slope of the graph of $f(t) = -t^3 + 2t$ at $t = 1$.

   Solution: We have $\frac{d}{dt}\left(-t^3 + 2t\right)\Big|_{t=1} = \left(-3t^2 + 2\right)\Big|_{t=1} = -1$.

3. Find an equation for the tangent line to $y = \frac{1}{2}x^2 + 3$ at $x = 1$. Use your calculator to view the function and the tangent line on the same screen. (Use the window $[-2, 4] \times [0, 8]$.)

   Solution: A point on the tangent line is the point on graph of $y = \frac{1}{2}x^2 + 3$ at $x = 1$, which is $(1, 3.5)$. The slope of the tangent line is $\frac{dy}{dx}\Big|_{x=1} = (x)\Big|_{x=1} = 1$. Therefore, the point-slope form of the tangent line is $y - 3.5 = 1(x - 1)$, and the slope-intercept form is $y = x + 2.5$.

4. Let $y = 3x^2$. Find Δy and dy when $x = 1$ and $\Delta x = 0.1$.

   Solution: The quantity Δy is the change in $y = 3x^2$ from $x = 1$ to $x + \Delta x = 1.1$. That is, $\Delta y = 3(1.1)^2 - 3(1)^2 = 0.63$.

   The quantity dy is the change in the tangent line to $y = 3x^2$ at $x = 1$ from $x = 1$ to $x + \Delta x = 1.1$. Since $\frac{dy}{dx} = 6x$, the differential $dy = 6x\,dx = 6(1)(0.1) = 0.6$.

5. The side length of a cube is measured as $x = 25$ cm, with possible error in measurement $\Delta x = \pm 0.01$ cm. The measurement is then used to calculate the volume $V = x^3$. Find the propagated and relative errors in the volume calculation.

   Solution: For propagated error, $\Delta V \approx dV = 3x^2 dx = 3(25)^2(\pm 0.01) = \pm 18.75$. That is, the volume calculation will be within ± 18.75 cm^3 of the actual value.

   For the relative error, $\frac{\Delta V}{V} \approx \frac{dV}{V} = \frac{3x^2 dx}{x^3} = \frac{3(25)^2(\pm 0.01)}{25^3} = \pm 0.0012$, which is within $\pm 0.12\%$ of actual value.