Examples 2.4 – Analyzing Cubic Functions

1. Let $y = -x^3 + 6x^2 - 12x + 72$.

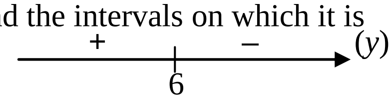
   (a) Find all roots (x-intercepts).

   Solution: In this case, we can group and factor:

   $$\left(-x^3 + 6x^2\right) + \left(-12x + 72\right) = -x^2(x-6) - 12(x-6) = (x-6)\left(-x^2 - 12\right)$$

   Since $(-x^2 - 12)$ is never zero, the only root is $x = 6$.

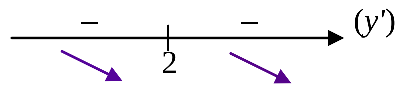
   (b) Determine the intervals on which the function is positive and the intervals on which it is negative.

   Solution: A sign test of y on either side of the root shows that y is positive on $(-\infty, 6)$ and negative on $(6, +\infty)$.

   (c) Find the derivative of y and use it to determine extrema, saddle points, and intervals of increase and decrease.

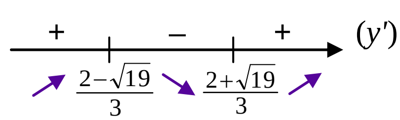
   Solution: The derivative of y is $y' = -3x^2 + 12x - 12 = -3(x^2 - 4x + 4) = -3(x-2)^2$, which only has one root, namely $x = 2$. In other words, y has a horizontal tangent at $x = 2$. A sign test of y' on either side of the root shows that y is decreasing on both sides of the horizontal tangent. We can conclude that y has a saddle point at $x = 2$, and y has no extrema.

2. Find the roots and extrema of $y = x^3 - 2x^2 - 5x + 6$.

   Solution: Grouping will not work here, but notice that $x = 1$ is a root. This means that $x - 1$ is a factor. We use long division (shown on the right) to get $y = (x-1)(x^2 - x - 6)$. By factoring the quotient, we obtain $y = (x-1)(x^2 - x - 6) = (x-1)(x-3)(x+2)$. Therefore, the roots are $x = -2$, 1, and 3. To find the extrema, we set the derivative $y' = 3x^2 - 4x - 5$ to zero By the quadratic formula, the roots of the derivative are $x = \frac{2 \pm \sqrt{19}}{3}$, and a sign test of the derivative shows that y has a local maximum of $y\left(\frac{2-\sqrt{19}}{3}\right) \approx 8.2$ at $x = \frac{2-\sqrt{19}}{3} \approx -0.8$, and a local minimum of $y\left(\frac{2+\sqrt{19}}{3}\right) \approx -4.1$ at $x = \frac{2+\sqrt{19}}{3} \approx 2.1$.

   $$\begin{array}{r l}
    & x^2 - x - 6 \\
   x-1 \,\big) & x^3 - 2x^2 - 5x + 6 \\
    & \underline{x^3 - x^2} \\
    & -x^2 - 5x \\
    & \underline{-x^2 + x} \\
    & -6x + 6 \\
    & \underline{-6x + 6} \\
    & 0
   \end{array}$$

3. Graph the functions from Parts 1 and 2 on your calculator and visually identify the characteristics we obtained above.