Lesson 2.3 – Definition and Properties of the Derivative

Recall the steps for finding the slope of the tangent line to the graph of f at the point $(x_0, f(x_0))$:

STEP 1: Pick $x \neq x_0$ on either side of x_0.

STEP 2: Simplify $\frac{\Delta y}{\Delta x} = \frac{f(x) - f(x_0)}{x - x_0}$, if possible. (1)

STEP 3: Let $x \to x_0$ from the left and right sides of x_0. (2)

The result, if one exists, is $f'(x_0)$. (3) This process is expressed formally using **limit notation:**

Limit definition of the derivative at a point: $f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$

Notes: (1) The simplification will not be much of an issue for simpler functions, but later we will need to use a different form for the limit expression. We derive this form below.

(2) The notation $x \to x_0$ implies that we are approaching x_0 from the left *and* the right. Because of this, the limit expression is called a **two-sided limit**. Later we will need to look at the limits from the left and right separately. These are called **one-sided limits**.

(3) For the derivative to exist, the one-sided limits must be equal to the same number. Again, this is not an issue right now since, for example, moving toward a point on a parabola will yield the same slope from either direction.

Since $\Delta x = x - x_0$, it follows that $x = x_0 + \Delta x$. If $x \to x_0$, then $\Delta x \to 0$. By substituting into the limit definition above, we get an alternate form for the derivative at a point, as well as a form for the derivative function itself:

Limit definition of the derivative at a point:

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

Limit definition of the derivative function:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

General properties of the derivative: (These properties will be verified in Activity 2.3.)
Assume all derivatives below exist.

1. **Constant Multiple Rule:** If k is a constant and $f(x) = k \cdot g(x)$, then $f'(x) = k \cdot g'(x)$.
   (In words: *The derivative of a constant multiple is the constant multiple of the derivative*.)

2. **Sum/Difference Rule:** If $f(x) = g(x) \pm h(x)$, then $f'(x) = g'(x) \pm h'(x)$
   (In words: *The derivative of a sum or difference is the sum or difference of the derivatives*.)